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Correlation functions for a periodic box–ball system

Jun Mada[1] **and Tetsuji Tokihiro**[2]

[1] College of Industrial Technology, Nihon University, 2-11-1 Shin-ei, Narashino, Chiba 275-8576, Japan
[2] Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Tokyo 153-8914, Japan

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Abstract
We investigate correlation functions in a periodic box–ball system. For the two-point functions of short distance, we give explicit formulae obtained by combinatorial methods. We give expressions for general N-point functions in terms of ultradiscrete theta functions.

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1. Introduction

Quantum integrable systems such as quantum integrable spin chains and solvable lattice models are systems whose Hamiltonians or transfer matrices can be diagonalized and for which eigenstates or free energies can be explicitly obtained [1]. However, to investigate physical properties of these systems, such as e.g. the linear response to external forces, we further need to evaluate correlation functions for these systems. This is one of the main problems in the field of quantum integrable systems and, in fact, obtaining correlation functions is fairly difficult even for the celebrated *XXZ* model or the six-vertex model [2].

A periodic box–ball system (PBBS) is a soliton cellular automaton obtained by ultradiscretizing the KdV equation [3, 4]. It can also be obtained at the $q \to 0$ limit of the generalized six-vertex model [5, 6]. Hence, from the viewpoint of quantum integrable lattice models, it is interesting and may actually give some new insights into the correlation functions of the vertex models themselves to obtain correlation functions of the PBBS. In this paper, we give expressions for N-point functions for the PBBS, using combinatorial methods and the solution for the PBBS expressed in terms of the ultradiscrete theta functions.

The PBBS can be defined as follows. Let $L \geqslant 3$ and let $\Omega_L = \{ f \mid f : [L] \to \{0, 1\}$ such that $\sharp f^{-1}(\{1\}) < L/2 \}$, where $[L] = \{1, 2, \ldots, L\}$. When $f \in \Omega_L$ is represented as a sequence of 0s and 1s, we write

$$f(1)f(2)\cdots f(L).$$

The mapping $T_L : \Omega_L \to \Omega_L$ is defined as follows (see figure 1).

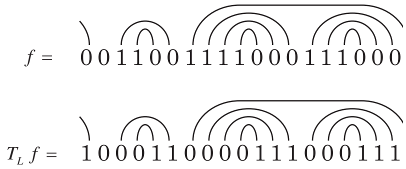

Figure 1. Definition of T_L for $f \in \Omega_L$.

1. In the sequence f, find a pair of positions n and $n+1$ such that $f(n) = 1$ and $f(n+1) = 0$, and mark them; repeat the same procedure until all such pairs are marked. Note that we always use the convention that the position n is defined in $[L]$, i.e. $n + L \equiv n$.
2. Skipping the marked positions we get a subsequence of f; for this subsequence, repeat the same process of marking positions, so that we get another marked subsequence.
3. Repeat step 2 until a subsequence consisting only of 0s is obtained. A typical situation is depicted in figure 1. After these preparatory processes, change all values at the marked positions simultaneously. One thus obtains the sequence $T_L f$.

Sometimes we shall write $T_L^t f$ for $\underbrace{T_L(\cdots(T_L(T_L}_{t} f)))$. The pair (Ω_L, T_L) is called a PBBS of length L [4, 7]. An element of Ω_L is called a state, and the mapping T_L the time evolution.

The conserved quantities of the PBBS are defined as follows. Let $Q_j(t)$ be the number 10 pairs in $T_L^t f$ marked at the jth step in the definition of the mapping T_L. Then we obtain a nonincreasing sequence of positive integers, $Q_j(t)$ $(j = 1, 2, \ldots, m)$. This sequence is conserved in time, that is,

$$Q_j(t) = Q_j(t+1) \equiv Q_j \qquad (j = 1, 2, \ldots, m).$$

For example, for f in figure 1, $(Q_1, Q_2, Q_3, Q_4) = (3, 3, 2, 1)$. As the sequence $(Q_1, Q_2, \ldots, Q_m)$ is nonincreasing, we can associate a Young's diagram with it by considering Q_j as the number of squares in the jth column of the diagram. The lengths of the rows are also weakly decreasing positive integers. Let the distinct row lengths be $P_1 > P_2 > \cdots > P_\ell$ and let n_j be the number of times the length P_j appears. The set $\{P_j, n_j\}_{j=1}^{\ell}$ is another expression for the conserved quantities of the PBBS.

An N-point function of the PBBS with M balls may be defined as follows.

$$\langle s_1, s_2, \ldots, s_N \rangle := \frac{1}{Z_H} \sum_{f \in \Omega_{L;M}} \mathrm{e}^{\sum_{k=1}^{L} H_k(f)} f(s_1) f(s_2) \cdots f(s_N)$$

where $\Omega_{L;M} := \{f \in \Omega_L \mid \sharp f^{-1}(\{1\}) = M\}$, $Z_H := \sum_{f \in \Omega_{L;M}} \mathrm{e}^{\sum_{k=1}^{L} H_k(f)}$ and $H_k(f)$ is the kth energy of the state f, which is proportional to the number of kth arc lines defined when determining the time evolution rule [4], or the kth conserved quantity of the PBBS [8]. (Note that $H_k(f)$ is essentially equal to the energy function for the transfer matrix of the crystal lattice models with $k+1$ states on a vertical link [5, 7].) Noticing the fact that $\Omega_{L;M} = \bigsqcup_Y \Omega_Y$,

$$\langle s_1, s_2, \ldots, s_N \rangle = \frac{1}{Z_H} \sum_{Y} \sum_{f \in \Omega_Y} \mathrm{e}^{\sum_{k=1}^{L} H_k(f)} f(s_1) f(s_2) \cdots f(s_N),$$

where Y are partitions of M corresponding to the conserved quantities of the PBBS (see section 2). Since, for $f_i \in \Omega_{Y_i}$ $(i = 1, 2)$, $\forall k$, $H_k(f_1) = H_k(f_2)$ $(k = 1, 2, 3, \ldots)$ implies

$Y_1 = Y_2$, and vice versa, by choosing a state f_Y in Ω_Y, we can write

$$\langle s_1, s_2, \ldots, s_N \rangle = \frac{1}{Z_H} \sum_Y \mathrm{e}^{\sum_{k=1}^{L} H_k(f_Y)} \sum_{f \in \Omega_Y} f(s_1) f(s_2) \cdots f(s_N).$$

Thus, to obtain correlation functions of PBBS, we only have to evaluate those on the set Ω_Y:

$$\langle s_1, s_2, \ldots, s_N \rangle_Y := \frac{1}{|\Omega_Y|} \sum_{f \in \Omega_Y} f(s_1) f(s_2) \cdots f(s_N). \tag{1}$$

We also point out that if we put $\forall k,\ \forall f,\ H_k(f) = 0$, N-point functions become trivial:

$$\langle s_1, s_2, \ldots, s_N \rangle = \frac{{}_{L-N}C_{M-N}}{{}_{L}C_M} = \frac{M(M-1)\cdots(M-N+1)}{L(L-1)\cdots(L-N+1)}.$$

In the following sections, we shall evaluate (1).

First we summarize some useful properties of the PBBS. We say that f has (or that there is) a 10-wall at position n if $f(n-1) = 1$ and $f(n) = 0$. Let the number of the 10-walls be s and let their positions be denoted by $a_1 > a_2 > \cdots > a_s$. Then, we have the following proposition.

Proposition 1 ([9]).

$$\left(T_L^t f\right)(n) = \eta_{n+1}^{t-1} - \eta_{n+1}^{t} - \eta_n^{t-1} + \eta_n^t,$$

$$\eta_n^t = \max_{\substack{m_i \in \mathbb{Z} \\ i \in [s]}} \left[\sum_{i=1}^{s} m_i (b_i + t W_i - n) - \sum_{i=1}^{\ell} \sum_{j=1}^{\ell} m_i \Xi_{ij} m_j \right], \tag{2}$$

$$b_i = a_i + \sum_{j=1}^{i-1} 2 \min\{W_i, W_j\} + W_i + \frac{Z_i}{2}, \tag{3}$$

$$\Xi_{ij} = \frac{Z_i}{2} \delta_{ij} + \min\{W_i, W_j\},$$

$$Z_i = L - \sum_{j=1}^{s} 2 \min\{W_i, W_j\},$$

where W_i denotes the amplitude of the 'soliton' corresponding to a_i obtained by the procedure explained in [9].

The set $\{W_i\}_{i=1}^{s}$ consists of quantities of the PBBS and η_n^t is the ultradiscrete theta function [10]. We shall use proposition 1 for determining N-point functions in section 3.

Next we introduce two procedures which are important in this paper. For a given $f \in \Omega_L$, a state $Ef = E(f)$ is defined to be

$$(Ef)(n) = \begin{cases} \begin{cases} f(n) & (1 \leqslant n \leqslant a_s - 2) \\ f(n+2k) & \begin{pmatrix} a_{s-k+1} - 2k + 1 \leqslant n \leqslant a_{s-k} - 2k - 2 \\ (k = 1, 2, \ldots, s-1) \end{pmatrix} \\ f(n+2s) & (a_1 - 2s + 1 \leqslant n \leqslant L - 2s) \end{cases} & (a_s > 1) \\[2ex] \begin{cases} f(n+1) & (1 \leqslant n \leqslant a_{s-1} - 3) \\ f(n+2k+1) & \begin{pmatrix} a_{s-k} - 2k \leqslant n \leqslant a_{s-k-1} - 2k - 3 \\ (k = 1, 2, \ldots, s-2) \end{pmatrix} \\ f(n+2s-1) & (a_1 - 2s + 1 \leqslant n \leqslant L - 2s). \end{cases} & (a_s = 1) \end{cases}$$

The mapping $E : \Omega_L \to \Omega_{L-2s}$ is called the 10-elimination. Ef is a subsequence of f obtained by eliminating all the 10-walls in f simultaneously. For example,

$$\begin{aligned}
f &= 001111\underline{10}0001111\underline{10}0000000111\underline{10}01\underline{10}0001111\underline{10}011\underline{10}000111\underline{10}000000000,\\
Ef &= 001111\quad 0001111\quad 0000000111\quad 01\quad 0001111\quad 011\quad 000111\quad 000000000\\
&= 001111000111100000001110100011110110001110000000000.
\end{aligned}$$

Its inverse process is called the 10-insertion, $I(j_1, j_2, \ldots, j_d) = I_2 \circ I_1(j_1, j_2, \ldots, j_d) : \Omega_L \to \Omega_{L+2(d+s)}$, where s is the number of 10-walls in $f \in \Omega_L$. The 10-insertion is defined as follows: shifting the origin if necessary, we can assume that $f(L) = 0$. For $\{j_1, j_2, \ldots, j_d\}$ $(1 < j_1 < j_2 < \cdots < j_d \leqslant L + d)$, the mapping $I_1(j_1, j_2, \ldots, j_d) : \Omega_L \to \Omega_{L+2d}$ is defined as

$$(I_1(j_1, j_2, \ldots, j_d)f)(n) = \begin{cases} 1 & (n = L + 2d - j_k - k + 1),\\ 0 & (n = L + 2d - j_k - k + 2),\\ f(n) & (1 \leqslant n \leqslant L + d - j_d),\\ f(n - 2(d - k + 1)) & (L + 2d - j_k - k + 3 \leqslant n \leqslant L + 2d - j_{k-1} - k + 1),\\ f(n - 2d) & (L + 2d - j_1 + 2 \leqslant n \leqslant L + 2d), \end{cases}$$

where $k \in [d]$; furthermore, $I_2 : \Omega_{L+2d} \to \Omega_{L+2(d+s)}$ is defined to be

$$(I_2 f')(n) = \begin{cases} 1 & (n = g_k + 2(s - k) + 1),\\ 0 & (n = g_k + 2(s - k) + 2),\\ f'(n) & (1 \leqslant n \leqslant g_s),\\ f'(n - 2(s - k + 1)) & (g_k + 2(s - k) + 3 \leqslant n \leqslant g_{k-1} + 2(s - k) + 2),\\ f'(n - 2s) & (g_1 + 2s + 1 \leqslant n \leqslant L + 2(d + s)), \end{cases}$$

where $k \in [s]$, $f' \equiv I_1(j_1, j_2, \ldots, j_d)f \in \Omega_{L+2d}$ and

$$\begin{aligned}
g'_k &= \max\{m \in [L + d] | m = a_k - 1 + \sharp\{r \in [d] | L + d - j_r + 1 < m\}\},\\
g_k &= g'_k + \sharp\{r \in [d] | L + d - j_r + 1 < g'_k\}.
\end{aligned} \tag{4}$$

For example,

$$f = 0011100111000000110100011100000000,$$

$$I_1(3, 11, 25)f = 001110011 * 1000001101000 * 1110000 * 00 \tag{5}$$

$$= 001110011\underline{10}1000001101000\underline{10}1110000\underline{10}00, \tag{6}$$

$$I(3, 11, 25)f = 00111\boxed{10}0011\underline{10}1\boxed{10}0000011\boxed{10}01\boxed{10}000\underline{10}111\boxed{10}0000\underline{10}00,$$

where $\underline{10}$ and $\boxed{10}$, respectively, denote the inserted 10 at $f \mapsto I_1(j_1, j_2, \ldots, j_d)f$ and $I_1(j_1, j_2, \ldots, j_d)f \mapsto I_2(I_1(j_1, j_2, \ldots, j_d)f)$.

In the above example, g'_k is the position of 1 found in the kth 10-wall in the sequence (5), and g_k is the position of 1 found in the kth 10-wall in the sequence (6). That is, $\{g'_k\}_{k=1}^{5} = \{27, 20, 18, 11, 5\}$ and $\{g_k\}_{k=1}^{5} = \{29, 21, 19, 12, 5\}$.

2. One- and two-point functions obtained by combinatorial methods

We assume that Y denoting the conserved quantities of $f \in \Omega_Y$ is the partition

$$(\underbrace{P_1, P_1, \ldots, P_1}_{n_1}, \underbrace{P_2, P_2, \ldots, P_2}_{n_2}, \ldots, \underbrace{P_\ell, P_\ell, \ldots, P_\ell}_{n_\ell}),$$

where $P_1 > P_2 > \cdots > P_\ell \geqslant 1$. Note that Y is a partition of M, i.e. $M = \sum_{i=1}^{\ell} n_i P_i$. As mentioned in section 1, we consider N-point functions (1) of the PBBS:

$$\langle s_1, s_2, \ldots, s_N \rangle_Y = \frac{1}{|\Omega_Y|} \sum_{f \in \Omega_Y} f(s_1) f(s_2) \cdots f(s_N).$$

The value of $|\Omega_Y|$ is already known.

Proposition 2 ([11]).

$$|\Omega_Y| = \frac{L}{L_0} \binom{L_0 + n_1 - 1}{n_1} \binom{L_1 + n_2 - 1}{n_2} \cdots \binom{L_{\ell-1} + n_\ell - 1}{n_\ell},$$

where $L_0 = L - 2M,\ L_i = L_0 + \sum_{j=1}^{i} 2n_j (P_j - P_{i+1})$ *and* $P_{\ell+1} = 0$.

Since the N-point function $\langle s_1, s_1 + d_1, \ldots, s_1 + d_{N-1} \rangle_Y$ does not depend on the specific site s_1 (because of translational symmetry), we denote

$$C_Y(d_1, d_2, \ldots, d_{N-1}) \equiv \langle s_1, s_1 + d_1, \ldots, s_1 + d_{N-1} \rangle_Y,$$

where $1 \leqslant d_1 < d_2 < \cdots < d_{N-1} < L$. Note that $C_Y(\emptyset)$ denotes the 1-point function $\langle s_1 \rangle_Y$.

Proposition 3.

$$C_Y(\emptyset) = \frac{M}{L}.$$

Proof. Since $\sum_{n=1}^{L} f(n) = M$,

$$LC_Y(\emptyset) = \sum_{s_1=1}^{L} \langle s_1 \rangle_Y = \frac{1}{|\Omega_Y|} \sum_{f \in \Omega_Y} \sum_{n=1}^{L} f(n) = \frac{1}{|\Omega_Y|} |\Omega_Y| M = M.$$

□

Next, we consider the 2-point functions.

Proposition 4.

$$C_Y(1) = \frac{M - s}{L},$$

where $s = \sum_{i=1}^{\ell} n_i$.

Proof. Since $\sum_{n=1}^{L} f(n) f(n+1) = M - s$,

$$LC_Y(1) = \frac{1}{|\Omega_Y|} \sum_{f \in \Omega_Y} \sum_{n=1}^{L} f(n) f(n+1) = M - s.$$

□

In order to investigate $C_Y(2)$, let us put

$$k_i := \begin{cases} n_j & (i = P_j), \\ 0 & \text{otherwise}, \end{cases}$$

$$\hat{k}_i := \sum_{j=i}^{P_1} k_j,$$

$$\tilde{L} := L - 2\hat{k}_1 \qquad (= L - 2s),$$

$$N_Y(2) := \sum_{i=3}^{P_1} k_i(i-2).$$

Note that k_i denotes the number of the rows of the conserved quantity Y with length i, and $\hat{k}_i$ denotes the number of the rows of the conserved quantity Y with length $\geqslant i$. Thus, $\hat{k}_1$ is the number of boxes in the first row of Y.

We also define

$$V_{f_0} := \{f \in \Omega_Y | Ef = f_0\},$$

$$G_2(f) := \sharp\{n \in [L] | f(n)f(n+2) = 1\}.$$

The following lemma is the key to evaluating $C_Y(2)$.

Lemma 1. *Let*

$$V_{f_0}^{(j)} := \{f \in V_{f_0} | G_2(f) = N_Y(2) + j\}.$$

Then, if $V_{f_0} \neq \phi$, $V_{f_0} = \bigsqcup_{j=0}^{k_1} V_{f_0}^{(j)}$ *and*

$$\left|V_{f_0}^{(k_1-j)}\right| = \frac{\nu_j}{k_1!}, \tag{7}$$

where

$$\nu_j := \left(\prod_{i=0}^{j-1}(\tilde{L} - 2\hat{k}_2 - i)\right)\left(\prod_{i=0}^{k_1+j-1}(2\hat{k}_2 + i)\right) \times \left(\sum_{\substack{0 \leqslant i_1 < \cdots < i_j < k_1+j-1 \\ i_r+1 < i_{r+1} \ (r \in [j-1])}} \prod_{h=1}^{j} \frac{1}{(2\hat{k}_2 + i_h)(2\hat{k}_2 + i_h + 1)}\right).$$

Proof. When $f \in V_{f_0}$, there exists a set of positive numbers $\{j_i\}_{i=1}^{k_1}$ $(1 < j_1 < j_2 < \cdots < j_{k_1} \leqslant \tilde{L} + k_1)$ such that

$$f = I(j_1, j_2, \ldots, j_{k_1}) f_0.$$

By examining the positions of 101 and 111, we find that

$$G_2(f) = N_Y(2) + \gamma + J,$$

where $\gamma = \gamma\left(f_0; \{j_i\}_{i=1}^{k_1}\right)$ is the number of $\underline{10}$s inserted into the positions adjacent to consecutive 1s, and $J = \sharp\left\{ i \in [d-1] \,\middle|\, j_i + 1 = j_{i+1} \right\}$ (see figure 2). For example,

$$f_0 = 001110000100110000,$$

f_0	00111000			
$G_2(f_0)$	1			
$f = I(k)f_0$	0010111100000 ($k=7$)	0011011100000 ($k=6$)	0011110100000 ($k=4$)	0011110000100 ($k=2$)
$G_2(f)$	3 ($\gamma = 1,\ J = 0$)	2 ($\gamma = 0,\ J = 0$)	3 ($\gamma = 1,\ J = 0$)	2 ($\gamma = 0,\ J = 0$)

Figure 2. Example $G_2(f)$, γ and J.

and $f = I(5, 6, 14, 15, 18)f_0$; then

$$f = 001111010001010001100011101010000 0$$
$$(= 00111\ 10\ 100010\ 10001\ 10\ 0011\ 10\ 10\ 100000\ ).$$

In this example, $k_1 = 5$, $\hat{k}_2 = 3$, $N_Y(2) = 3$, $\gamma = 2$ and $J = 2$. Since $0 \leqslant \gamma + J \leqslant k_1$, we have the decomposition $V_{f_0} = \bigsqcup_{j=0}^{k_1} V_{f_0}^{(j)}$.

To know $\left|V_{f_0}^{(j)}\right|$, we only have to count the number of states with $\gamma + J = j$.

For $k_1 = 1$, $|V_{f_0}| = \tilde{L}$. Since there are $\hat{k}_2$ sets of consecutive 1s, $2\hat{k}_2$ states have $\gamma + J = 1$ $(\gamma = 1,\ J = 0)$ and the other $\tilde{L} - 2\hat{k}_2$ states have $\gamma + J = 0$ $(\gamma = 0,\ J = 0)$.

For $k_1 = 2$, as was seen in case $k_1 = 1$, there are $2\hat{k}_2$ positions at which $\gamma + J$ can be increased by 1. If one 10 pair is inserted in one of these positions, then there are $2\hat{k}_2 + 1$ positions for the other pair to increase $\gamma + J$ by 1 and $\tilde{L} - 2\hat{k}_2$ positions not to increase it. On the other hand, if one 10 pair is inserted at one of the $\tilde{L} - 2\hat{k}_2$ non-increasing positions, then there are $2\hat{k}_2 + 2$ positions for the other pair to increase $\gamma + J$ by 1 and $\tilde{L} - 2\hat{k}_2 - 1$ positions not to increase it. Hence, considering duplication of insertion, there are $(2\hat{k}_2)(2\hat{k}_2 + 1)/2!$ states with $\gamma + J = 2$, $\left[(2\hat{k}_2)(\tilde{L} - 2\hat{k}_2) + (\tilde{L} - 2\hat{k}_2)(2\hat{k}_2 + 2)\right]/2!$ states with $\gamma + J = 1$ and $(\tilde{L} - 2\hat{k}_2)(\tilde{L} - 2\hat{k}_2 - 1)/2!$ states with $\gamma + J = 0$.

In general, we can proceed in a similar manner and, referring to the chart in figure 3, we obtain (7). □

Proposition 5.

$$C_Y(2) = \frac{\displaystyle\sum_{j=0}^{k_1} \nu_j \left(\sum_{i=3}^{P_1} k_i(i-2) + (k_1 - j) \right)}{\displaystyle L \sum_{j=0}^{k_1} \nu_j}.$$

Proof. From lemma 1, we see that if $V_{f_0} \neq \phi$,

$$\sum_{f \in V_{f_0}} \sum_{n=1}^{L} f(n)f(n+2) = \sum_{j=0}^{k_1} \frac{\nu_j}{k_1!} \Big(N_Y(2) + (k_1 - j) \Big)$$

and

$$\left|V_{f_0}\right| = \sum_{j=0}^{k_1} \frac{\nu_j}{k_1!}.$$

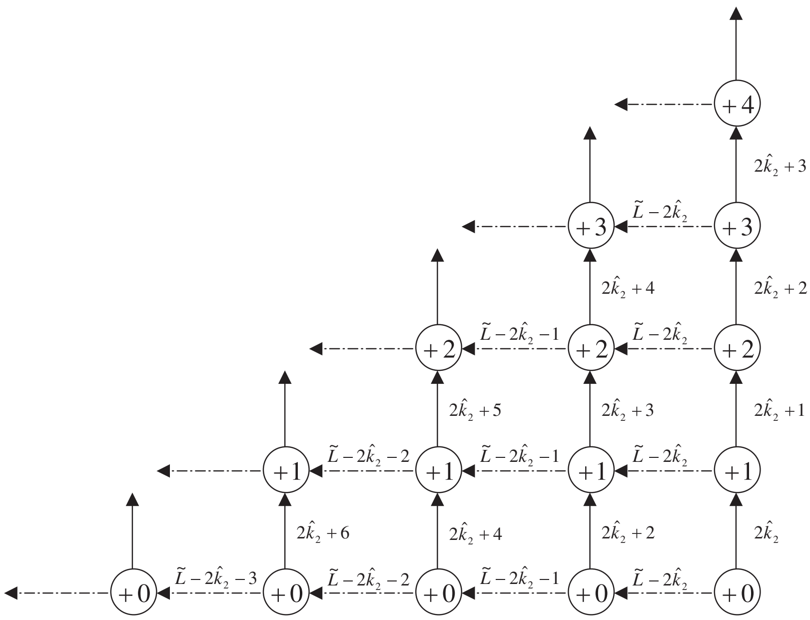

Figure 3. A chart corresponding to $\gamma + J$ in the proof of lemma 1.

Since the right-hand side of the last equation does not depend on f_0, and since any state $f \in \Omega_Y$ belongs to some V_{f_0}, we obtain

$$LC_Y(2) = \frac{1}{|\Omega_Y|} \sum_{f \in \Omega_Y} \sum_{n=1}^{L} f(n) f(n+2) = \frac{\sum_{j=0}^{k_1} \nu_j \Big(N_Y(2) + (k_1 - j) \Big)}{\sum_{j=0}^{k_1} \nu_j}.$$

□

For $C_Y(d)$ $(d \geqslant 3)$, we can use similar arguments based on elementary combinatorics. However, the expressions become more and more complicated when the difference d increases. Instead, in the next section, we shall use proposition 1 to obtain expressions for general N-point functions.

3. *N*-point correlation functions for the PBBS

Let the state f_0 and the set $\mathcal{X}_Y \subset \mathbb{Z}_+^{n_1} \times \mathbb{Z}_+^{n_2} \times \cdots \times \mathbb{Z}_+^{n_\ell}$ $(= \mathbb{Z}_+^s)$ be

$$f_0 = \underbrace{000 \cdots 00}_{L_0},$$

and

$$\mathcal{X}_Y := \left\{ \{x_i(k)\}_{i=1,k=1}^{\ell,\ \ n_i} \;\middle|\; \begin{array}{r} 1 < x_i(1) < x_i(2) < \cdots < x_i(n_i) \leqslant L_{i-1} + n_i \\ (i = 1, 2, \ldots, \ell) \end{array} \right\}. \tag{8}$$

We define the state f_X recursively as

$$f_j := \underbrace{I(\emptyset)\cdots I(\emptyset)}_{P_j - P_{j+1} - 1} I(X_j) f_{j-1} \quad (j = 1, 2, \ldots, \ell),$$

$$f_X := f_\ell,$$

where $X_j = \{x_j(k)\}_{k=1}^{n_j} \subset X \in \mathcal{X}_Y$. Note that, from the definition of a 10-insertion, $I(\emptyset)$ is the procedure needed to insert $\boxed{10}$s between 10:

$$f = 0011100111000001101000111000000,$$

$$I(\emptyset)f = 00111\boxed{10}00111\boxed{10}0000011\boxed{10}01\boxed{10}000111\boxed{10}000000,$$

and $f_X \in \Omega_Y$ by construction. We also define $\tilde{\Omega}_Y$ by

$$\tilde{\Omega}_Y := \{f_X | X \in \mathcal{X}_Y\}.$$

Lemma 2.

$$\langle s_1, s_2, \ldots, s_N\rangle_Y = \frac{1}{L|\tilde{\Omega}_Y|} \sum_{f\in\tilde{\Omega}_Y} \sum_{k=1}^{L} f(k+s_1) f(k+s_2)\cdots f(k+s_N). \tag{9}$$

Proof. By virtue of the definition of f_X, $\tilde{\Omega}_Y$ is the set of states with conserved quantities Y and the last entry of the 10-sequence is one of the 0s that are not marked in the time evolution rule, i.e. $f_X(L) = (T_L f_X)(L) = 0$. By defining the shift operator S by $(Sf)(n) := f(n+1)$, and $(S^k f) := S(S^{k-1} f)$ $(k = 1, 2, \ldots)$ with $S^0 f := f$ and for sets

$$S^k\tilde{\Omega}_Y := \{S^k f_X | X \in \mathcal{X}_Y\} \qquad (k = 1, 2, \ldots, L),$$

we find

$$\forall f \in \Omega_Y, \qquad \sharp\{k | f \in S^k\tilde{\Omega}_Y\ (k = 1, 2, \ldots, L)\} = L_0.$$

Note that $S^L f = f$. Since $|\Omega_Y| = \frac{L}{L_0}|\tilde{\Omega}_Y|$,

$$\begin{aligned}\langle s_1, s_2, \ldots, s_N\rangle_Y &= \frac{1}{|\Omega_Y|} \sum_{f\in\Omega_Y} f(s_1) f(s_2)\cdots f(s_N)\\ &= \frac{1}{L|\tilde{\Omega}_Y|} \sum_{k=1}^{L} \sum_{f\in S^k\tilde{\Omega}_Y} f(s_1) f(s_2)\cdots f(s_N)\\ &= \frac{1}{L|\tilde{\Omega}_Y|} \sum_{k=1}^{L} \sum_{f\in\tilde{\Omega}_Y} f(s_1+k) f(s_2+k)\cdots f(s_N+k).\end{aligned}$$

Thus, we obtain (9). □

Proposition 6. *For $X \in \mathcal{X}_Y$, f_X is explicitly given as*

$$f_X(n) = u_n^0(X),$$

where

$$u_n^t(X) := \eta_{n+1}^{t-1}(X) - \eta_{n+1}^t(X) - \eta_n^{t-1}(X) + \eta_n^t(X),$$

$$\eta_n^t(X) := \max_{\substack{m_{ij}\in\mathbb{Z},\\ i\in[\ell];\ j\in[n_i]}} \left[\sum_{i=1}^{\ell}\sum_{k=1}^{n_i} m_{ik}\left(tP_i - n - x_i(k) + L + k + 1 + \frac{Z_i}{2}\right) - \sum_{i=1}^{\ell}\sum_{k=1}^{n_i}\sum_{j=1}^{\ell}\sum_{h=1}^{n_j} m_{ik}\,\Xi_{ikjh} m_{jh} \right], \tag{10}$$

$$\Xi_{ikjh} := \frac{Z_i}{2}\delta_{ij}\delta_{kh} + P_{\max[i,j]},$$
$$Z_i := L - 2\left(P_i \sum_{j=1}^{i} n_j + \sum_{j=i+1}^{\ell} n_j P_j\right).$$

Proof. From proposition 1, f_X is determined by the parameters W_n and a_n $(n = 1, 2, \ldots, s)$. Here W_n is the amplitude of the nth soliton and a_n is its position, i.e. the position of the nth 10-wall, counting from the right. From the definition of the position and the amplitude of a soliton, it follows that both can be determined from 10-insertions. Because of the way f_X was constructed, the set $\{x_j(k)\}_{k=1}^{n_j}$ corresponds to the position of n_j solitons with amplitude P_j, though it does not directly give their position. Hereafter, we shall refer to a soliton with amplitude P as a P-soliton. By considering the relation between the position of a soliton and 10-insertions, we find that the position of the kth P_j-soliton counting from the right is $L - x_j^{(\ell)}(k) + 2$, where $x_j^{(\ell)}(k)$ is determined recursively: we define $x_j^{(i)}(k)$ $(i \in [\,\ell\,],\ j \in [\,i\,],\ k \in [n_j])$ as

$$x_j^{(i)}(k) := x_j(k) + (P_j - P_{i+1})(2\beta_j(k) + 2k - 1) + \sum_{s=j+1}^{i} 2(P_s - P_{i+1})\alpha_j^{(s)}(k) - k + 1,$$

where

$$\alpha_j^{(i)}(k) := \sharp\big\{r \in [n_i]\,\big|\,L_{i-1} + n_i - x_i(r) + 1 > g_j^{(i)}(k)\big\},$$

$$\beta_1(k) := 0, \qquad \beta_i(k) := \sum_{s=1}^{i-1} \sharp\big\{r \in [n_s]\,\big|\,g_s^{(i)}(r) \geqslant L_{i-1} + n_i - x_i(k) + 1\big\},$$

$$g_j^{(i)}(k) := \max\left\{m \in [L_{i-1} + n_i]\,\middle|\,\begin{array}{l} m = L_{i-1} - x_j^{(i-1)}(k) + 1 \\ \quad + \sharp\{r \in [n_i] | L_{i-1} + n_i - x_i(r) + 1 \leqslant m\}\end{array}\right\}.$$

Note that $x_j^{(i)}(1) < x_j^{(i)}(2) < \cdots < x_j^{(i)}(n_j)$. We show an example in appendix A.

Recalling the fact that $\sharp\{\,r \in [d]\mid L+d-j_r+1 < g'_k\,\}$ in (4) is the number of inserted $\underline{10}$s, on the left of the kth soliton (here we do not count the inserted $\underline{10}$s as solitons), the concrete meaning of these variables becomes clear: $\alpha_j^{(i)}(k)$ denotes the number of P_i-solitons on the right of the kth P_j-soliton, and $\beta_j(k)$ denotes the number of solitons with amplitudes greater than P_j, to the right of the kth P_j-soliton.

Since $\big\{L - x_j^{(\ell)}(k) + 2\big\}_{j=1,k=1}^{\ell,n_j}$ is the complete set of positions of the solitons, there exists a one-to-one mapping $\rho : \{\,(j,k)\mid j \in [\,\ell\,],\ k \in [n_j]\,\} \to [\,s\,]$ such that

$$a_{\rho(j,k)} = L - x_j^{(\ell)}(k) + 2.$$

From these recursion relations, we have

$$x_j^{(\ell)}(k) = x_j(k) + P_j(2\beta_j(k) + 2k - 1) + \sum_{i=j+1}^{\ell} 2P_i\alpha_j^{(i)}(k) - k + 1$$
$$= x_j(k) + 2\left\{P_j\big(\beta_j(k) + (k-1)\big) + \sum_{i=j+1}^{\ell} P_i\alpha_j^{(i)}(k)\right\} + P_j - k + 1.$$

Since the position of the kth P_j-soliton is $a_{\rho(j,k)}$, $W_{\rho(j,k)} = P_j$ and the set of amplitudes of the solitons on the right of the kth P_j-soliton is nothing but $\{W_h\}_{h=1}^{\rho(j,k)-1}$. From the definition of $\alpha_j^{(i)}(k)$, $\beta_j(k)$,

$$\alpha_j^{(i)}(k) = \sharp\{h \in [\rho(j,k)-1] | W_h = P_i\},$$
$$\beta_j(k) = \sharp\{h \in [\rho(j,k)-1] | W_h > P_j\},$$

and

$$\sharp\{h \in [\rho(j,k)-1] | W_h = P_j\} = k-1.$$

Thus, we obtain

$$x_j^{(\ell)}(k) = x_j(k) + \sum_{h=1}^{\rho(j,k)-1} 2\min\{W_{\rho(j,k)}, W_h\} + W_{\rho(j,k)} - k + 1.$$

Therefore, we find a concrete expression of $a_{\rho(j,k)}$, and (10) is immediately obtained from (2) and (3). □

From lemma 2 and proposition 6, we immediately obtain the following theorem.

Theorem 1. *Let $\mathcal{X}_Y$ be the set defined in (8). We then have*

$$C_Y(d_1, d_2, \ldots, d_{N-1}) = \frac{1}{L|\mathcal{X}_Y|} \sum_{X\in\mathcal{X}_Y} \sum_{n=1}^{L} u_n(X) \prod_{i=1}^{N-1} u_{n+d_i}(X),$$

for $u_n(X) \equiv u_n^0(X)$, as given in (10).

4. Concluding remarks

In this paper, we investigated correlation functions for the PBBS and obtained explicit forms for 1-point and 2-point functions at short distances. We also give expressions in terms of ultradiscrete theta functions for general N-point functions. Investigating their asymptotic properties and clarifying the relation to correlation functions for quantum integrable systems are problems that will be addressed in the future.

Finally, we comment on the time averages of quantities in the PBBS. The time average

$$C_f(d_1, d_2, \ldots, d_{N-1}) = \frac{1}{L|\mathcal{T}_f|} \sum_{t=1}^{\mathcal{T}_f} \sum_{n=1}^{L} \left(T_L^t f\right)(n) \prod_{j=1}^{N-1} \left(T_L^t f\right)(n+d_j),$$

where $\mathcal{T}_f$, the fundamental cycle of $f \in \Omega_L$, depends not only on the conserved quantities of the state but, in general, also on the initial state f itself. For example, the conserved quantities of the states $f_1 = 0100100$ and $f_2 = 0101000$ are the same, but $C_{f_1}(3) = \frac{1}{7}$ and $C_{f_2}(3) = 0$. Hence, in general, $C_f(d_1, d_2, \ldots, d_{N-1}) \neq C_Y(d_1, d_2, \ldots, d_{N-1})$ even for $f \in \Omega_Y$. Note that for the 1-point function $C_f(\emptyset)$, we can easily show that

$$\forall f \in \Omega_Y, \qquad C_f(\emptyset) = C_Y(\emptyset) = \frac{M}{L}.$$

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Appendix A. Example to determine the position of solitons

Let $L = 33$, $(P_1, n_1) = (3, 2)$, $(P_2, n_2) = (2, 3)$ and $(P_3, n_3) = (1, 2)$. Since $L_0 = 5$, $L_1 = 9$, $L_2 = 19$, we suppose $\{x_i(k)\}_{i=1,k=1}^{\ell, \quad n_i} \in \mathcal{X}_Y$ to be

$x_1(1) = 3,\ x_1(2) = 6;\ x_2(1) = 2,\ x_2(2) = 5,\ x_2(3) = 9;\ x_3(1) = 12,\ x_3(2) = 17.$

In this case, by successive 10-insertions, we obtain the state

$$011100100110001001110011000011000.$$

Then, we see from [9] that

$$\{(a_k, W_k)\}_{k=1}^{7} = \{(31, 2), (25, 3), (21, 2), (16, 1), (12, 2), (8, 1), (5, 3)\}.$$

On the other hand, according to the algorithm in the proof of proposition 6, the variables are determined as follows.

(i) The variables $x_1^{(1)}(k)$ are determined from the initial data:

$$\begin{aligned} x_1^{(1)}(1) &= x_1(1) + (P_1 - P_2)(2\beta_1(1) + 2\cdot 1 - 1) - 1 + 1 \\ &= 3 + (3-2)(2\cdot 0 + 2\cdot 1 - 1) - 1 + 1 = 4, \\ x_1^{(1)}(2) &= x_1(2) + (P_1 - P_2)(2\beta_1(2) + 2\cdot 2 - 1) - 2 + 1 \\ &= 6 + (3-2)(2\cdot 0 + 2\cdot 2 - 1) - 2 + 1 = 8. \end{aligned}$$

(ii) To determine the variables $x_j^{(2)}(k)$, we calculate $g_1^{(2)}(k)$, $\alpha_j^{(2)}(k)$ and $\beta_2(k)$:

$$g_1^{(2)}(1) = \max\left\{ m \in [L_1 + n_2] \,\middle|\, \begin{array}{l} m = L_1 - x_1^{(1)}(1) + 1 \\ \qquad + \sharp\{r \in [n_2] | L_1 + n_2 - x_2(r) + 1 \leqslant m\} \end{array} \right\}$$

$$= \max\left\{ m \in [12] \,\middle|\, m = 6 + \sharp\left\{ r \in [3] \,\middle|\, \begin{array}{ll} 11 & (r=1) \\ 8 & (r=2) \\ 4 & (r=3) \end{array} \right\} \leqslant m \right\} = 8,$$

$$g_1^{(2)}(2) = \max\left\{ m \in [12] \,\middle|\, m = 2 + \sharp\left\{ r \in [3] \,\middle|\, \begin{array}{ll} 11 & (r=1) \\ 8 & (r=2) \\ 4 & (r=3) \end{array} \right\} \leqslant m \right\} = 2;$$

$$\alpha_1^{(2)}(1) = \sharp\{r \in [n_2] | L_1 + n_2 - x_2(r) + 1 > g_1^{(2)}(1)\}$$

$$= \sharp\left\{ r \in [3] \,\middle|\, \begin{array}{ll} 11 & (r=1) \\ 8 & (r=2) \\ 4 & (r=3) \end{array} \right\} > 8 \right\} = 1,$$

$$\alpha_1^{(2)}(2) = \sharp\left\{ r \in [3] \,\middle|\, \begin{array}{ll} 11 & (r=1) \\ 8 & (r=2) \\ 4 & (r=3) \end{array} \right\} > 2 \right\} = 3;$$

$$\beta_2(1) = \sharp\{r \in [n_1] | g_1^{(2)}(r) > L_1 + n_2 - x_2(1) + 1\}$$

$$= \sharp\left\{ r \in [2] \,\middle|\, \begin{array}{ll} 8 & (r=1) \\ 2 & (r=2) \end{array} \right\} \geqslant 11 \right\} = 0,$$

$$\beta_2(2) = \sharp\left\{r \in [2] \left| \begin{matrix} 8 & (r=1) \\ 2 & (r=2) \end{matrix} \right\} \geqslant 8 \right\} = 1,$$

$$\beta_2(3) = \sharp\left\{r \in [2] \left| \begin{matrix} 8 & (r=1) \\ 2 & (r=2) \end{matrix} \right\} \geqslant 4 \right\} = 1.$$

Then, we have

$$\begin{aligned} x_1^{(2)}(1) &= x_1(1) + (P_1 - P_3)(2\beta_1(1) + 2\cdot 1 - 1) + 2(P_2 - P_3)\alpha_1^{(2)}(1) - 1 + 1 \\ &= 3 + (3-1)(2\cdot 0 + 2\cdot 1 - 1) + 2(2-1)\cdot 1 - 1 + 1 = 7, \end{aligned}$$

$$x_1^{(2)}(2) = 17;$$

$$\begin{aligned} x_2^{(2)}(1) &= x_2(1) + (P_2 - P_3)(2\beta_2(1) + 2\cdot 1 - 1) - 1 + 1 \\ &= 2 + (2-1)(2\cdot 0 + 2\cdot 1 - 1) - 1 + 1 = 3, \end{aligned}$$

$$x_2^{(2)}(2) = 9, \qquad x_2^{(2)}(3) = 14.$$

(iii) To determine the variables $x_j^{(3)}(k)$, we calculate $g_j^{(3)}(k)$, $\alpha_j^{(3)}(k)$ and $\beta_3(k)$:

$$g_1^{(3)}(1) = \max\left\{m \in [21] \left| m = 13 + \sharp\left\{r \in [2] \left| \begin{matrix} 10 & (r=1) \\ 5 & (r=2) \end{matrix} \right\} \leqslant m \right\}\right.\right\} = 15,$$

$$g_1^{(3)}(2) = \max\left\{m \in [21] \left| m = 3 + \sharp\left\{r \in [2] \left| \begin{matrix} 10 & (r=1) \\ 5 & (r=2) \end{matrix} \right\} \leqslant m \right\}\right.\right\} = 3;$$

$$g_2^{(3)}(1) = \max\left\{m \in [21] \left| m = 17 + \sharp\left\{r \in [2] \left| \begin{matrix} 10 & (r=1) \\ 5 & (r=2) \end{matrix} \right\} \leqslant m \right\}\right.\right\} = 19,$$

$$g_2^{(3)}(2) = \max\left\{m \in [21] \left| m = 11 + \sharp\left\{r \in [2] \left| \begin{matrix} 10 & (r=1) \\ 5 & (r=2) \end{matrix} \right\} \leqslant m \right\}\right.\right\} = 13,$$

$$g_2^{(3)}(3) = \max\left\{m \in [21] \left| m = 6 + \sharp\left\{r \in [2] \left| \begin{matrix} 10 & (r=1) \\ 5 & (r=2) \end{matrix} \right\} \leqslant m \right\}\right.\right\} = 7;$$

$$\alpha_1^{(3)}(1) = \sharp\left\{r \in [2] \left| \begin{matrix} 10 & (r=1) \\ 5 & (r=2) \end{matrix} \right\} > 15 \right\} = 0,$$

$$\alpha_1^{(3)}(2) = \sharp\left\{r \in [2] \left| \begin{matrix} 10 & (r=1) \\ 5 & (r=2) \end{matrix} \right\} > 3 \right\} = 2;$$

$$\alpha_2^{(3)}(1) = \sharp\left\{r \in [2] \left| \begin{matrix} 10 & (r=1) \\ 5 & (r=2) \end{matrix} \right\} > 19 \right\} = 0,$$

$$\alpha_2^{(3)}(2) = \sharp\left\{r \in [2] \left| \begin{matrix} 10 & (r=1) \\ 5 & (r=2) \end{matrix} \right\} > 13 \right\} = 0,$$

$$\alpha_2^{(3)}(3) = \sharp\left\{r \in [2] \left| \begin{matrix} 10 & (r=1) \\ 5 & (r=2) \end{matrix} \right\} > 7 \right\} = 1;$$

$$\beta_3(1) = \sharp\left\{r \in [2] \,\middle|\, \begin{matrix} 15 & (r=1) \\ 3 & (r=2) \end{matrix} \right\} \geqslant 10 \Bigg\} + \sharp\left\{ r \in [3] \,\middle|\, \begin{matrix} 19 & (r=1) \\ 13 & (r=2) \\ 7 & (r=3) \end{matrix} \right\} \geqslant 10 \Bigg\} = 3,$$

$$\beta_3(2) = \sharp\left\{r \in [2] \,\middle|\, \begin{matrix} 15 & (r=1) \\ 3 & (r=2) \end{matrix} \right\} \geqslant 5 \Bigg\} + \sharp\left\{ r \in [3] \,\middle|\, \begin{matrix} 19 & (r=1) \\ 13 & (r=2) \\ 7 & (r=3) \end{matrix} \right\} \geqslant 5 \Bigg\} = 4.$$

Then, we have

$$\begin{aligned}
x_1^{(3)}(1) &= x_1(1) + P_1(2\beta_1(1) + 2\cdot 1 - 1) + \sum_{s=2}^{3} 2P_s\alpha_1^{(s)}(1) - 1 + 1 \\
&= 3 + 3(2\cdot 0 + 2\cdot 1 - 1) + 2(2\cdot 1 + 1\cdot 0) - 1 + 1 = 10, \\
x_1^{(3)}(2) &= 30; \\
x_2^{(3)}(1) &= x_2(1) + P_2(2\beta_2(1) + 2\cdot 1 - 1) + 2P_3\alpha_2^{(3)}(1) - 1 + 1 \\
&= 2 + 2(2\cdot 0 + 2\cdot 1 - 1) + 2\cdot 1\cdot 0 - 1 + 1 = 4, \\
x_2^{(3)}(2) &= 14, \quad x_2^{(3)}(3) = 23; \\
x_3^{(3)}(1) &= x_3(1) + P_3(2\beta_3(1) + 2\cdot 1 - 1) - 1 + 1 \\
&= 12 + 1\cdot(2\cdot 3 + 2\cdot 1 - 1) - 1 + 1 = 19, \\
x_3^{(3)}(2) &= 27.
\end{aligned}$$

(iv) Finally, we obtain the position of solitons:

$$L - x_j^{(3)}(k) + 2 = \begin{cases} 25 & (j=1,\ k=1), \\ 5 & (j=1,\ k=2); \\ 31 & (j=2,\ k=1), \\ 21 & (j=2,\ k=2), \\ 12 & (j=2,\ k=3); \\ 16 & (j=3,\ k=1), \\ 8 & (j=3,\ k=2). \end{cases}$$

Therefore, we confirm that $L - x_j^{(\ell)}(k) + 2$ is the position of the kth soliton among the solitons with amplitude P_j.

Appendix B. Example of values for the correlation function

From theorem 1, we obtain the following examples.

(a) $L = 12;\ P_1 = 3,\ n_1 = 1;\ P_2 = 1,\ n_2 = 2:$

$C_Y(\emptyset) = \frac{5}{12}, \quad C_Y(1) = \frac{1}{6}, \quad C_Y(2) = \frac{13}{84}, \quad C_Y(3) = \frac{19}{126}, \quad C_Y(1,2) = \frac{5}{84};$

(b) $L = 14;\ P_1 = 2,\ n_1 = 2;\ P_2 = 1,\ n_2 = 2:$

$C_Y(\emptyset) = \frac{3}{7}, \quad C_Y(1) = \frac{1}{7}, \quad C_Y(2) = \frac{5}{49}, \quad C_Y(3) = \frac{82}{441}, \quad C_Y(1,2) = 0;$

(c) $L = 14;\ P_1 = 3,\ n_1 = 1;\ P_2 = 1,\ n_2 = 3:$

$$C_Y(\emptyset) = \tfrac{3}{7},\quad C_Y(1) = \tfrac{1}{7},\quad C_Y(2) = \tfrac{5}{28},\quad C_Y(3) = \tfrac{69}{392},\quad C_Y(1,2) = \tfrac{5}{112};$$

(d) $L = 14;\ P_1 = 3,\ n_1 = 1;\ P_2 = 2,\ n_2 = 1;\ P_3 = 1,\ n_3 = 1:$

$$C_Y(\emptyset) = \tfrac{3}{7},\quad C_Y(1) = \tfrac{3}{14},\quad C_Y(2) = \tfrac{3}{28},\quad C_Y(3) = \tfrac{13}{112},\quad C_Y(1,2) = \tfrac{1}{16}.$$

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